Applied Mathematics Preliminary Exam
(Fall 2018)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**
2. **Use separate sheets for the solution of each problem.**

Problem 1 Suppose a bead, of mass m, slides frictionlessly on a hoop of radius R. If we then spin the hoop at constant angular velocity ω about an axis parallel to the force of gravity (see Fig. 1), the bead obeys the following non-linear second order differential equation

$$\frac{d^2\theta}{dt^2} - \omega^2 \sin(\theta)\cos(\theta) + \frac{g}{R}\sin(\theta) = 0$$

where g is the acceleration of gravity, $\theta(t)$ is the bead's angular position on the hoop (with $\theta = 0$ being at the bottom), and t is time.

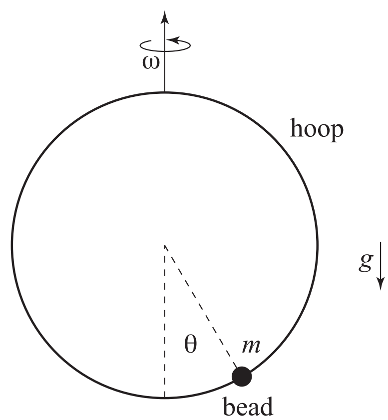

Figure 1:

a) Use non-dimensionalization to show that the qualitative behavior of the system is defined by a single non-dimensional parameter.

b) Find all fixed points, determine their stability and classify them as a function of that parameter.

c) Sketch a bifurcation plot (i.e., sketch the fixed points as a function of the parameter, indicate the stability of the fixed points, and label any bifurcations that occur). Use the Lyapunov definition of stability for this part.

It may or may not be useful to know that the energy of the system can be written as

$$E = mg\left(R - R\cos(\theta)\right) + \frac{m}{2}\left(R^2\sin^2(\theta) + R^2\left(\frac{d\theta}{dt}\right)^2\right)$$

The Lyapunov definition of stability is that a fixed point is stable if all trajectories starting sufficiently close to the fixed point remain within an arbitrarily small distance of the fixed point.

Problem 2 A solid box, with sides of unequal length, obeys Euler's equations when tossed in the air:

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} = \mathbf{0}$$

where, for simplicity, we neglect gravity. In this equation, $\boldsymbol{I}$ is the inertia tensor (defined below) and $\boldsymbol{\omega}$ is the angular velocity vector.

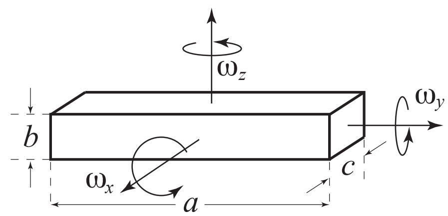

Figure 2:

For the box above (Fig. 2), with length a, height b, and width c, the inertia tensor (in Cartesian coordinates) is

$$\boldsymbol{I} = \begin{bmatrix} \frac{m}{12}(a^2+b^2) & 0 & 0 \\ 0 & \frac{m}{12}(c^2+b^2) & 0 \\ 0 & 0 & \frac{m}{12}(a^2+c^2) \end{bmatrix}$$

and the corresponding angular velocity vector is

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

For the following, assume $a > c > b$, and $||\boldsymbol{\omega}|| = 1$.

a) Find all fixed point(s).

b) Use linear stability analysis to classify the fixed point(s), i.e., stable node, unstable node, center, stable spiral, unstable spiral, saddle.

Problem 3 Find a planar curve $(x, y) = (x(t), y(t))$ that minimizes the following functional:

$$I = \int_0^1 m\left(\frac{\dot{x}^2 + \dot{y}^2}{2} - g\,y\right) \mathrm{d}t,$$

where m, g are positive constants, $(x(0), y(0)) = (0, 0)$, and $(x(1), y(1)) = (a, 0)$.

[Physically, this is a problem to find a trajectory of a projectile of mass m that starts at $(0, 0)$ and hits at $(a, 0)$ at time $t = 1$ under gravity.]

Problem 4 Consider the *Regular Sturm-Liouville Problem* on the unit interval $[0, 1]$:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\,x^2} + \lambda f = 0, \quad f(0) = 0, f(1) + f'(1) = 0.$$

a) Find the eigenvalues and eigenfunctions of this RSL system.

[Hint: Those eigenvalues are the solutions of some transcendental (also known as *secular*) equation.]

b) Expand the constant function 1 on $[0, 1]$ into the series of the eigenfunctions obtained in Part (a).

Problem 5 The modified Bessel function $I_n(x)$ for n an integer has the integral representation

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} \exp\left(x\cos\theta\right)\cos(n\theta)\,d\theta.$$

Find the leading order asymptotic expansion for $I_n(x)$ as $x \to \infty$. You may find the following integrals useful:

$$\int_{-\infty}^{\infty} \exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}},\ a > 0; \qquad \Gamma(x) = \int_0^{\infty} t^{x-1}\mathrm{e}^{-t}\,dt.$$

Problem 6

a) Show that the all of the solutions to

$$\ddot{u} + u + \epsilon u^3 = 0, \quad \epsilon \geq 0$$

are periodic in time.

[Hint: One could show that all nontrivial trajectories in the phase plane are closed curves.]

b) For $\epsilon = 0$, the period of the oscillation is 2π. Find the leading ϵ-dependent correction to the period in the limit of of small ϵ for solutions that pass through the point

$$u(t_0) = A, \quad \dot{u}(t_0)(0) = 0,$$

where $t = t_0$ is some time.

End of the exam.